The Riesz gases and their phase diagram in dimension 1

On systems of interacting particles, IHES

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What are Riesz gases $(d \geq 1)$?

What are formally Riesz gases ?

- Infinite # of point-like particles x_j in $\mathbb{R}^d$

- The particles x_j interact through the **Riesz potential** :

$$V_s(x) = \begin{cases} |x|^{-s} & s > 0 \\ -\ln|x| & s = 0 \\ -|x|^{-s} & -2 < s < 0 \end{cases}$$

- Sign is chosen to make V_s **repulsive** : particles want to get far away from one another

- Energy of j_0-th particle in configuration $\{x_j : j \in \mathbb{N}\}$: $\sum_{j \neq j_0} V_s(x_j - x_{j_0})$

- **Short-range** $s > d$: summable series for typical configurations

- **Long-range** $s \leq d$: divergent series and **renormalization** is needed

- We will also consider **positive temperature** $T > 0$

Why do we care about Riesz gases ?

Riesz gases embodies several specific and important cases

Coulomb gases

$s = 1$ is the Coulomb potential in dimension 3

$s = d - 2$ is the « Coulomb » potential in dimension $d \geq 1$, *i.e.* solution to $-\Delta_{\mathbb{R}^d} V_s = \delta_0$

$s = d - 1$ is the « Coulomb » potential in dimension $d + 1$ restricted to hyperplans $-\Delta_{\mathbb{R}^{d+1}} V_s = \delta_{\mathbb{R}^d \times \{0\}}$

Log gases

$s = 0$ in dimension $d \in \{1,2\}$ are very important models which are believed to be sort of « universal »

Examples — *Gaussian ensemble in RMT, Ginzburg-Landau vortices, zeros of ζ function …*

Other examples

$s = 3$ in dimensions $d \in \{1,2,3\}$ is dipole-dipole interaction

…

The short-range case $s > d$

Definition of the Riesz gases for $s > d \quad (T = 0)$

The statistical physics way (thermodynamic limit)

Q° — *How to define such an infinite system ?*

Fix a domain $\Omega \subset \mathbb{R}^d$ and place N particles inside Ω : $\quad E_s(\Omega, N) := \min_{x_1,\ldots,x_N \in \Omega} \left\{ \sum_{1 \le i < j \le N} V_s(x_i - x_j) \right\}$

Thermodynamic limit : $\Omega \nearrow \mathbb{R}^d$ and $N \to \infty$ and keep average density fixed $|\Omega|/N := \rho > 0$

Energy per unit volume : $\quad e(s, \rho) := \lim_{\Omega \nearrow \mathbb{R}^d \, N \to \infty} \dfrac{E_s(\Omega, N)}{|\Omega|}$ with $\dfrac{|\Omega|}{N} = \rho$

Remark — By homogeneity, it holds that $E_s(\Omega, N) = \lambda^s E_s(\lambda\Omega, N)$. Choosing $\lambda = \rho^{1/d}$, one may assume $\rho = 1$.

Theorem — *The limit exists for $s > d$, for $\Omega = N^{1/d}\omega$ for $|\omega| = 1$ and $|\partial\omega| = 0$, rewrites by scaling as $e(s, \rho) = e(s)\rho^{1+s/d}$ and is independent of the shape of ω*

[Ruelle, *Statistical Mechanics : Rigorous results*, '99]

Definition of the Riesz gases for $s > d \quad (T > 0)$

The statistical physics way (thermodynamic limit)

Energy at temperature $T > 0$: $$F_s(\Omega, N, T) := \min_{\mathbb{P} \subset \Omega^N} \left\{ \int_{\Omega^N} \left(\sum_{1 \le i < j \le N} V_s(x_i - x_j) \right) d\mathbb{P} + T\,\mathrm{Ent}(\mathbb{P}) \right\}$$

Gibbs variational principle : $$\mathbb{P}_{s,\Omega,N,T} = Z_s(\Omega, N, T)^{-1} \exp\left(-\frac{1}{T} \sum_{1 \le i < j \le N} V_s(x_i - x_j) \right)$$

Partition function $\displaystyle \int_{\Omega^N} \exp\left(-\frac{1}{T} \sum_{1 \le i < j \le N} V_s(x_i - x_j) \right)$

Free energy per unit volume : $$f(s, \rho) := \lim_{\Omega \nearrow \mathbb{R}^d \; N \to \infty} \frac{F_s(\Omega, N, T)}{|\Omega|} \text{ with } \frac{|\Omega|}{N} = \rho$$

Theorem — *The limit exists for $s > d$ for any sequence of domains $\Omega = N^{1/d}\omega$ where $|\omega| = 1$ and $|\partial\omega| = 0$. By scaling, we have that the limit rewrites as $f(s, T, \rho) = f(s, T)\rho^{1+s/d}$ and that it is independent of the shape of ω.* [Ruelle, *Statistical Mechanics : Rigorous results*, '99]

Idea of proof $s > d$

Prove **subadditivity** of the energy $E_s(\Omega_1 \cup \Omega_2, N_1 + N_2) \leq E_s(\Omega_1, N_1) + E_s(\Omega_2, N_2)$

For hypercubes :

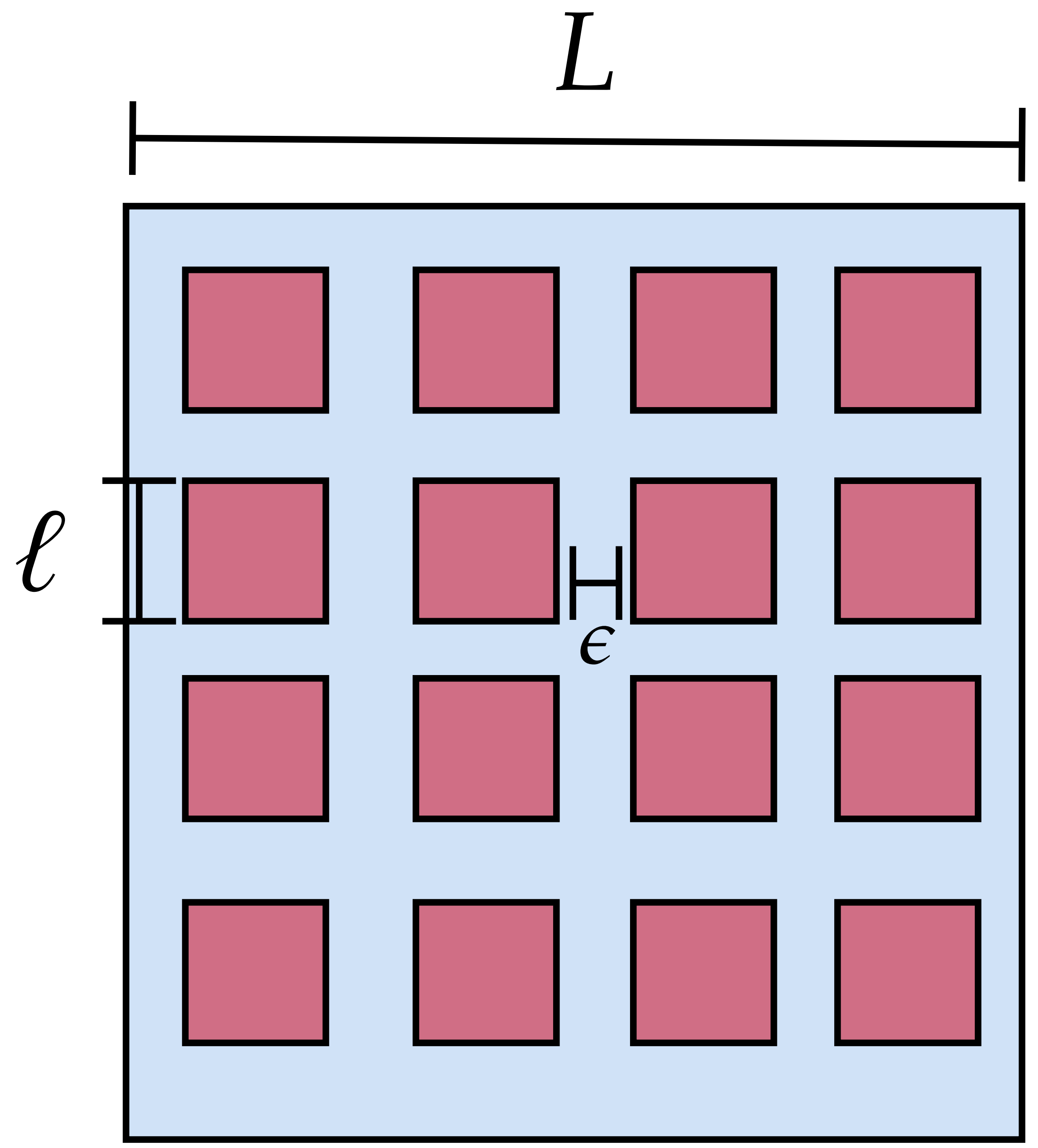
Take a big hypercube $C_L = L\omega$ for $\omega = [-1,1]^d$

Tile C_L into L^d/ℓ^d smaller hypercubes C_ℓ with small corridors of size $\epsilon \ll \ell \ll L$

In each small cubes, place ℓ^d particles so as to minimize energy in C_ℓ

This gives a trial-state for the big cube and thus an upper bound

$$E_s(C_L, L^d) \leq \frac{L^d}{\ell^d} E(C_\ell, \ell^d) + \underbrace{\text{Interaction between small cubes}}_{I_s(L,\ell,\epsilon)}$$



By integrability of V_s in the **short-range case** $s > d$, one can choose ϵ big enough to ensure that the interactions are small enough, so that

$$\limsup_{L\to\infty} \frac{E_s(C_L, L^d)}{L^d} \leq \liminf_{\ell\to\infty} \frac{E_s(C_\ell, \ell^d)}{\ell^d} \quad \blacksquare$$

For general domains Ω : Tile the domain with hypercubes etc…

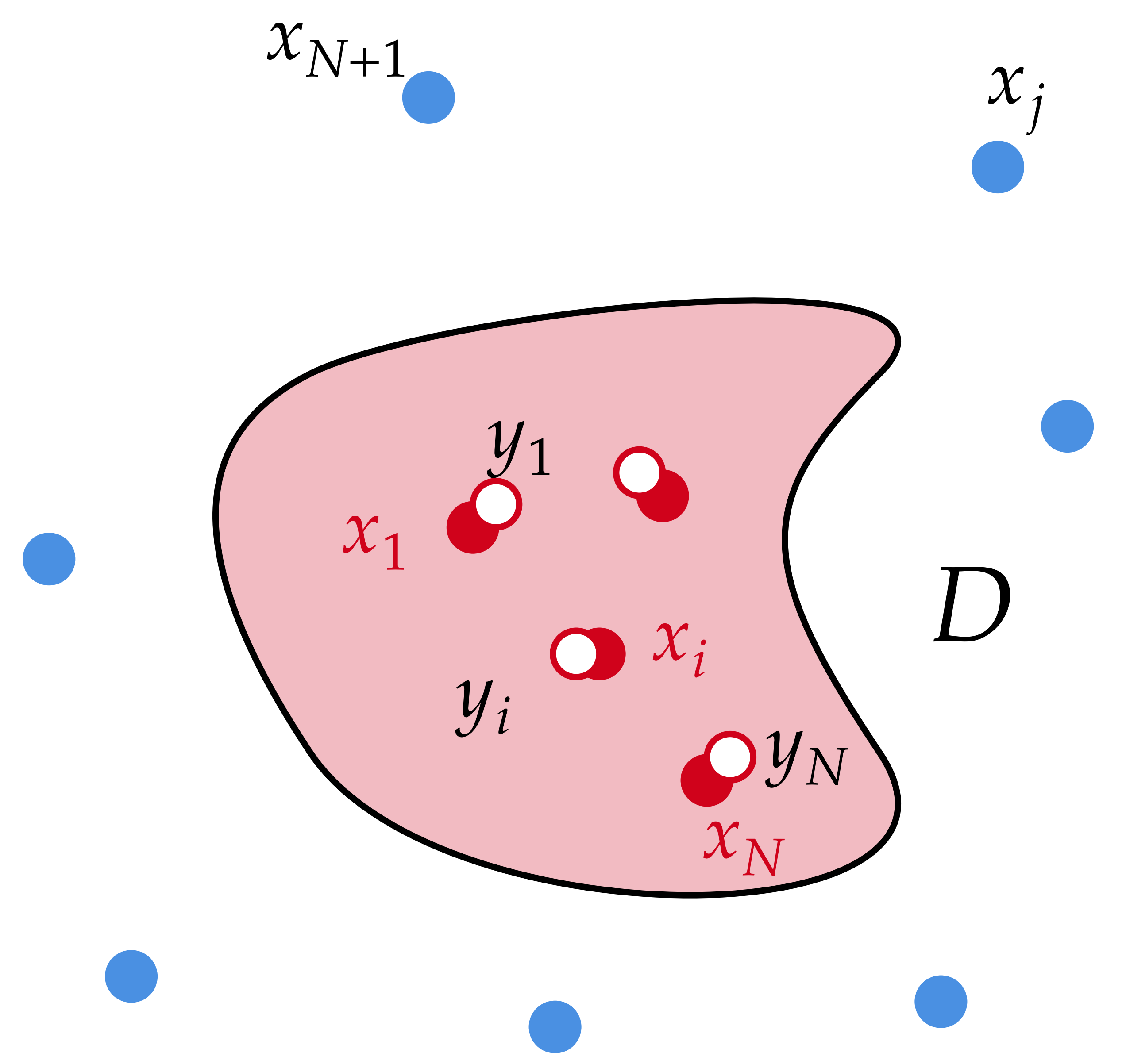
Infinite equilibrium configuration ($T = 0$)

So far, we only looked at the **(free) energy** in the thermodynamic limit. What about the points ?

The problem is that the energy of an infinite configuration $\{x_j : j \in \mathbb{N}\}$ is obviously infinite

Definition — *An infinite equilibrium configuration $\{x_j : j \in \mathbb{N}\}$ is a configuration which minimizes locally the energy in the sense that for any bounded $D \subset \mathbb{R}^d$ if we let $x_1, \ldots, x_N$ be the particles inside of D, then*

$$x_1, \ldots, x_N = \arg \min_{y_1,\ldots,y_N \in D} \left\{ \sum_{1 \le i < j \le N} V_s(y_i - y_j) + \sum_{i=1}^{N} \sum_{j=N+1}^{\infty} V_s(y_i - x_j) \right\}$$



A Riesz point process at T = 0 is then defined as a ***point process*** *on $\mathbb{R}^d$ which concentrates over such equilibrium configurations.*

Theorem — *In the short-range case $s > d$, such a point process exists.* [Lewin, JMP '22]

Idea of proof — *The minimizers of $E_s(\Omega, N)$ do not cluster or leave big holes (i.e. number of points is locally bounded above and below). Take the thermodynamic limit and use compactness.*

Infinite equilibrium configuration ($T > 0$)

In the positive temperature case $T > 0$, a similar definition holds

Definition — *A Riesz point process at temperature T > 0 is a* ***point process*** *on* $\mathbb{R}^d$ *such that for any bounded* $D \subset \mathbb{R}^d$ *the conditional law* $\mathbb{P}_{s,T,D}$ *of the point process given that the number of points in* D *is* N *and the positions* $\{x_k\}_{k=N+1}^{\infty}$ *of the particles outside of* D *verifies*

$$\mathbb{P}_{s,T,D} = \arg\min_{\mathbb{P} \subset D^N} \left\{ \int_{D^N} \left(\sum_{1 \le i < j \le N} V_s(y_i - y_j) + \sum_{i=1}^{N} \sum_{k=N+1}^{\infty} V_s(y_i - x_k) \right) d\mathbb{P} + T\, \mathrm{Ent}(P) \right\}$$

This is called the (canonical) Dobrushin—Lanford—Ruelle (DLR) equations.

Remark — According to **Gibbs variational principle** :

$$\mathbb{P}_{s,T,D}(y_1, \ldots, y_N) = Z(s,T,D)^{-1} \exp\left(-\frac{1}{T} \left(\sum_{1 \le i < j \le N} V_s(y_i - y_j) + \sum_{i=1}^{N} \sum_{k=N+1}^{\infty} V_s(y_i - x_k) \right) \right)$$

Theorem — *In the short-range case* $s > d$*, such a point process exists for all* $T > 0$ [Lewin, JMP '22]

The long-range case $s < d$

Long-range case $s \leq d$

$$E_s(\Omega, N) := \min_{x_1,\ldots,x_N \in \Omega} \left\{ \sum_{1 \leq i < j \leq N} V_s(x_i - x_j) \right\}$$

When $s \leq d$, particles will accumulate on the boundary $\partial\Omega$

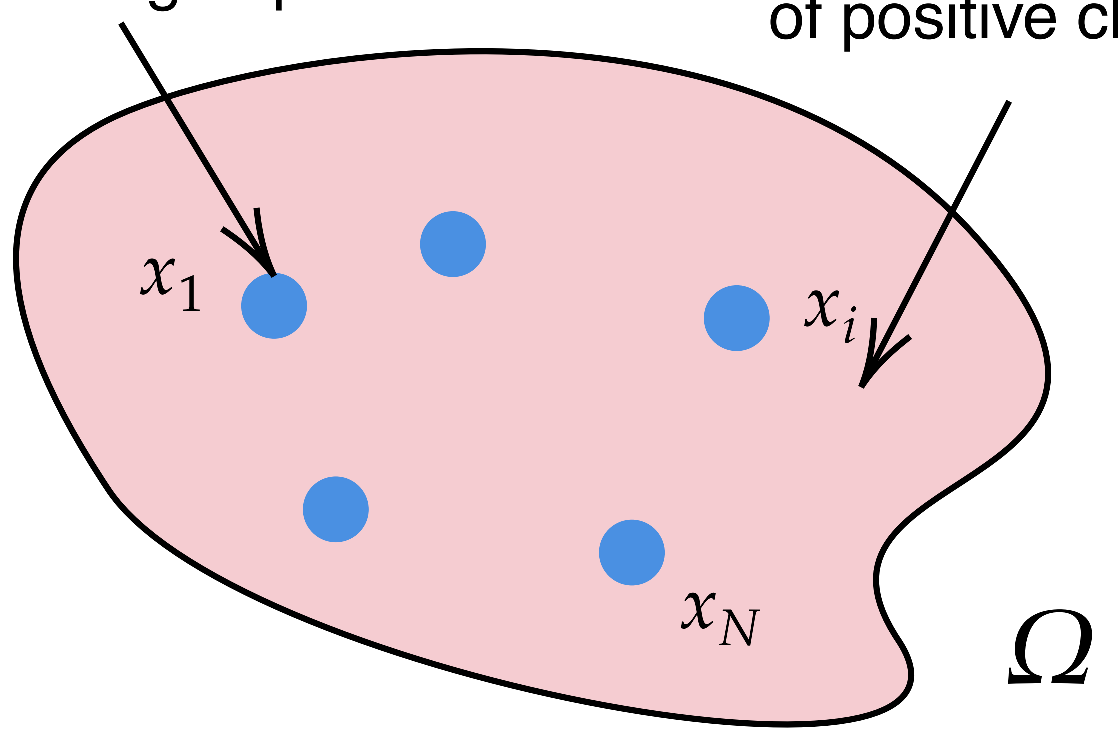
In fact, for $s \leq d - 2$ particles will be exactly on the boundary $\partial\Omega$ by superharmonicity $-\Delta V_s \geq 0$

Theorem — *For $s \leq d$ and for $\Omega = N^{1/d}\omega$ where $|\omega| = 1$ and $|\partial\omega| = 0$*

$$E_s(\Omega, N) \sim \frac{N^{2-s}}{2} \min_{\nu} \iint_{\omega\times\omega} \frac{d\nu(x)d\nu(y)}{|x-y|^s} = \underbrace{C(\omega)}_{>0} N^{2-s}$$

[Choquet '52, Messer—Spohn '82]

Renormalisation



We add a **uniform compensating background** of opposite charge on Ω

$$E_s(\Omega, N) := \min_{x_1,\ldots,x_N \in \Omega} \left\{ \sum_{1 \leq i < j \leq N} V_s(x_i - x_j) - \sum_{i=1}^{N} \rho_b \int_{\Omega} V_s(x_i - y)dy + \frac{\rho_b^2}{2} \iint_{\Omega\times\Omega} V_s(z - y)dzdy \right\}$$

Remark — For $s \leq 0$, one needs to assume neutrality $\rho_b = \rho$

In physics, this is called **Jellium** (atomic lattice, core of stars etc.)

Existence of thermodynamic limit of (free) energy

Theorem — *For* $d \geq 1$ *and* $\max(0,d-2) \leq s < d$ *or also* $s = -1$ *for* $d = 1$*, then for any sequence* $\Omega = N^{1/d}\omega$ *for* $|\omega| = 1$ *and* $|\partial\omega| = 0$*, we have existence of the thermodynamic limit for the (free) energy per unit volume*

$$\lim_{\Omega \nearrow \mathbb{R}^d \, N \to \infty} \frac{E_s(\Omega, N)}{|\Omega|} = e(s, \rho), \qquad \lim_{\Omega \nearrow \mathbb{R}^d \, N \to \infty} \frac{F_s(\Omega, N, T)}{|\Omega|} = f(s, \rho, T)$$

For Coulomb $s = 1$ in dimension $d = 3$: [Lieb—Narnhofer, '75]

Generalization to Coulomb $s = d - 2$ for $d \geq 1$: [Sari—Merlini, '76]

1D Coulomb $s = -1$: [Kunz, '74]

Other values of s : works of [Serfaty, Leblé, Petrache, Rougerie, Sandier…]

Why is the proof more complicated for long-range ?

If we do the same as in the short-range case (*i.e.* tile our domain in smaller domains), the interaction between the smaller domains will be harder to make negligible due to the non-integrability of V_s

Existence of the point processes in the long-range case $s \leq d$

The existence of a Riesz point process for $T \geq 0$ in the long-range case is **much more complicated** and many results are still completely **open**.

In the short-range case $s > d$ the existence of a Riesz point process $T \geq 0$ follows by proving local bounds on the number of particles (no cluster and big holes). In the long-range case $s \leq d$, this is not sufficient !

The difficulty is to define the potential generated inside a bounded domain $D \subset \mathbb{R}^d$ by the particles outside in order to prove DLR equations because of the non-integrability of V_s when $s \leq d$.

Theorems — *We have existence of a Riesz point process for the values:*

- $d - 1 < s < d$ *for* $T > 0$ *and* $d \geq 1$ [Dereudre & Vasseur '21]
- $s = 0$ *and* $d = 1$ *for* $T > 0$ [Dereudre, Hardy, Leblé & Maïda '21]
- $0 < s < d$ *in* $d = 1,2$ *and* $d - 2 \leq s < d$ *for* $d \geq 3$ *at* $T = 0$ [Lewin '22]

Remark — There are other characterization point processes than DLR equations and which may be easier to work with : **BBKGY hierarchy**, **Kirkwood-Salsburg** (KS) equations, **Kubo-Martin-Schwinger** (KMS) condition…

Phase transition in Riesz gases in dimension 1 : the (un)known.

What are phase transitions ?

There are many ways to define what a **phase transition** is…

The simplest definition is whether or not the Riesz point process at $T \geq 0$ is **unique**.

In the thermodynamics limit ($N < \infty$), the model is invariant under both translations and rotations, and it is conjectured that the set of Riesz point process $\mathscr{R}_{s,T}$ for a given s and $T \geq 0$ is either reduced to a point (*i.e.* no phase transition) or should be given by the convex hull of few simples point processes Γ_i obtained by isometries.

At zero temperature, the **cristallisation conjecture** states that $\mathscr{R}_{s,0}$ should be obtained by a periodic lattice Γ_1 (*e.g.* $\Gamma_1 = \rho^{-1}\mathbb{Z}$ in $d = 1$) and its image under translations and rotations.

At very high $T \gg 1$ we expect $\mathscr{R}_{s,T}$ is reduced to a point invariant under isometries (fluid)

Theorem (Mermin—Wagner for Riesz gases) — *In dimension $d \in \{1,2\}$ for $s > d$ and $T > 0$, the equilibrium states in $\mathscr{R}_{s,T}$ are all translation-invariant (i.e. fluids). For $d = 1$ and $s > 2$, $\mathscr{R}_{s,T}$ is furthermore reduced to a single point (i.e. no phase transition)* [Fröhlich & Pfister, '81 & '86] [Papangelou, '87]

Old « theorem » in physics which basically says ***there are no symmetry breaking in small dimensions***

Phase diagram of 1D Riesz gases in s and T

Coulomb gas $s = -1$:

Completely integrable model [Kunz '74, Aizenman & Martin '80]

It is crystalized at all temperature $T \geq 0$!

Log gas $s = 0$:

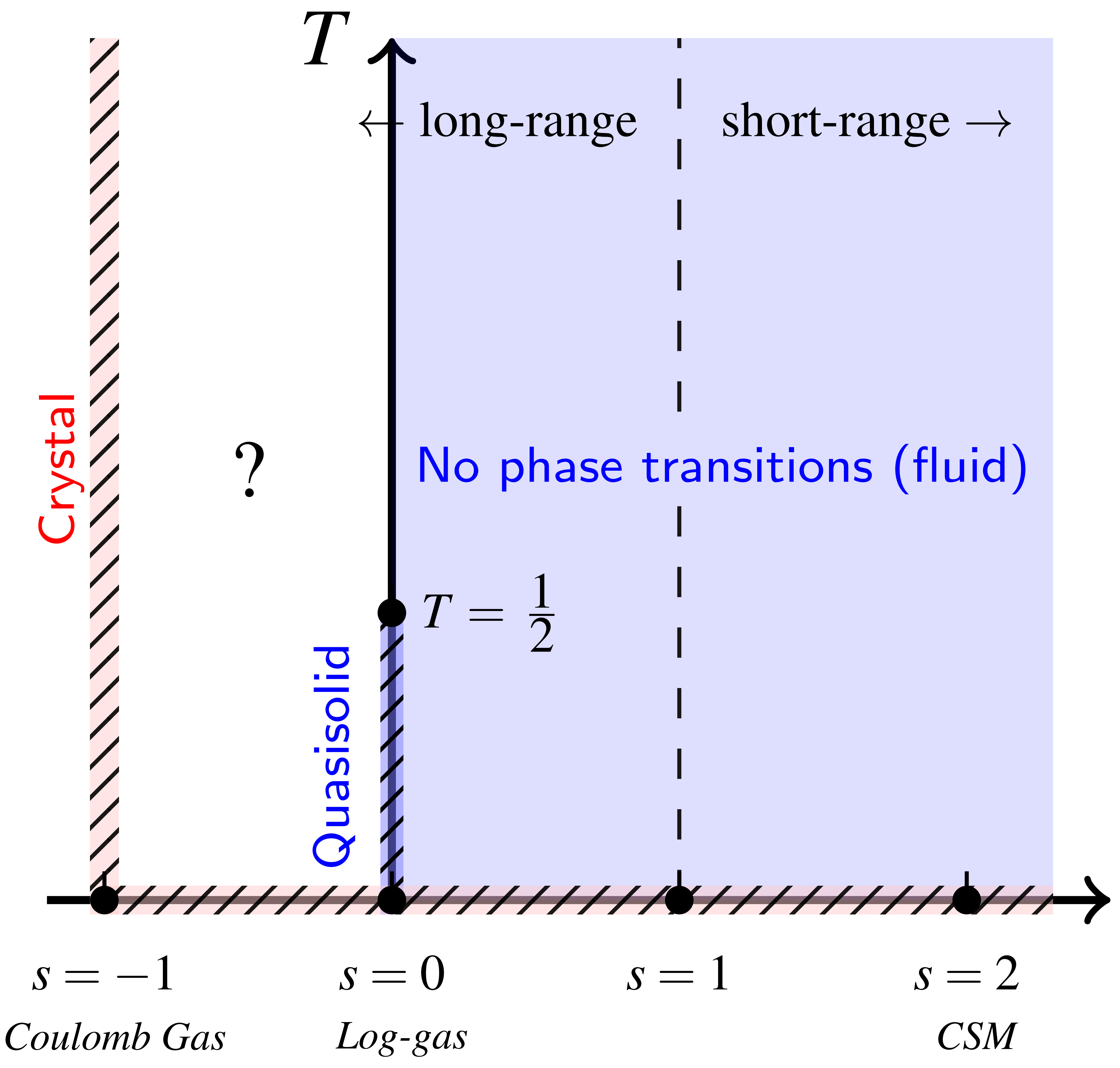
Integrable model linked to Gaussian ensembles in RMT

Crystalized for $T = 0$, and point process is believed to be unique & translation invariant for $T > 0$ (fluid)
[Serfaty & Sandier '15, Leblé '15, Erbar, Huesmann, Leblé '21]

There is a « phase transition » of a special kind at $T = 1/2$
[Forrester '84 & '10]

Case $s > 0$:

Believed that the point process is unique & translation for $T > 0$ (known for $s > 2$) and cristalized for $T = 0$



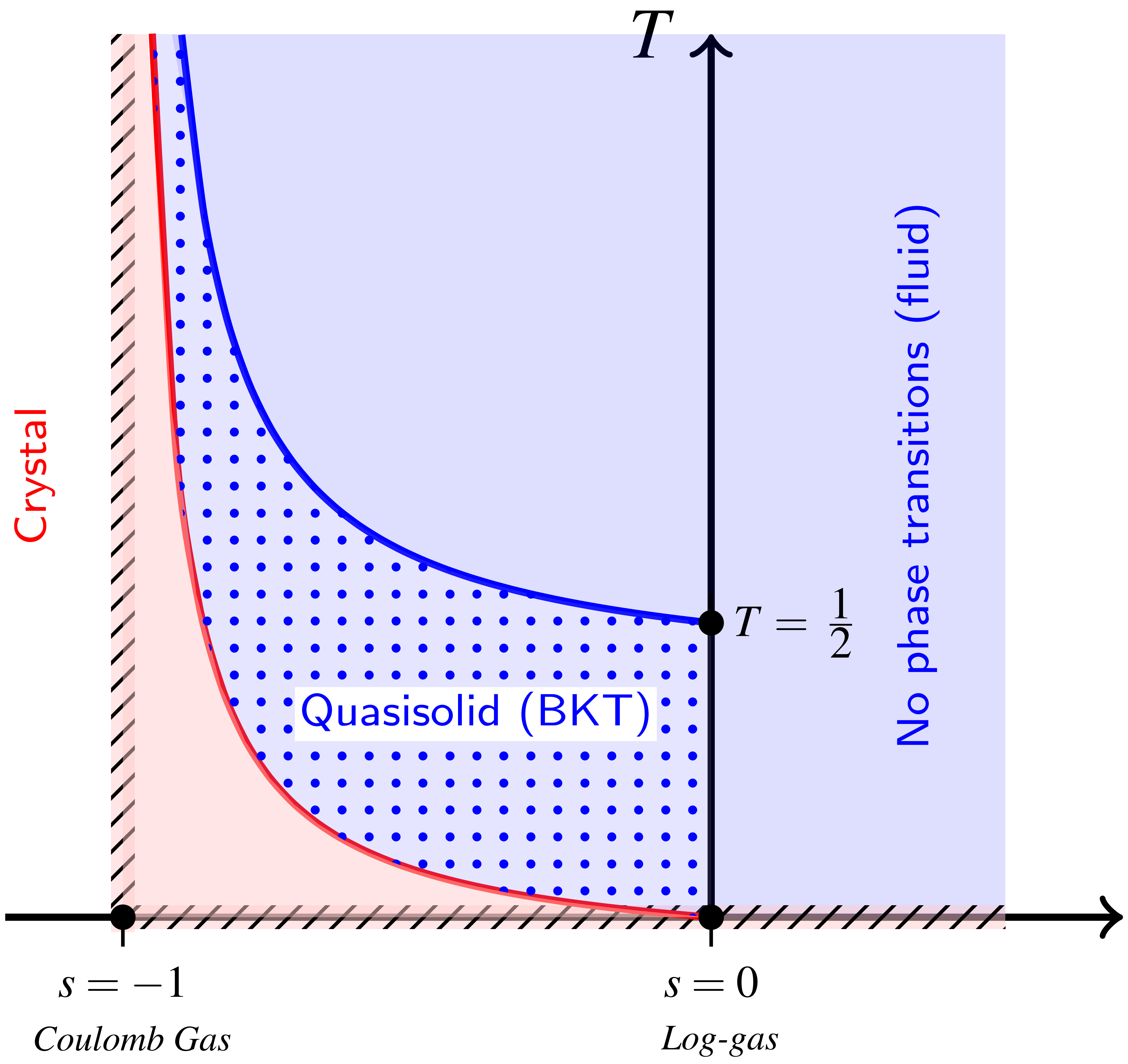
Q° — What is happening for $-1 < s < 0$?

What is happening for $-1 < s < 0$?

« *An interesting question is to fill the gaps in the picture and understand, in particular, if there exists a smooth transition curve to a periodic crystal in the region $-1 < s < 0$. A similar question concerns the BKT transition.* » [Lewin, '22]

Numerics seem to confirm this intuition:

Phase transitions in one-dimensional Riesz gases with long-range interaction, L '23



Monte Carlo simulations for $-1 < s < 0$ (I)

We simulate the Riesz gas with $N \gg 1$ using **MCMC**

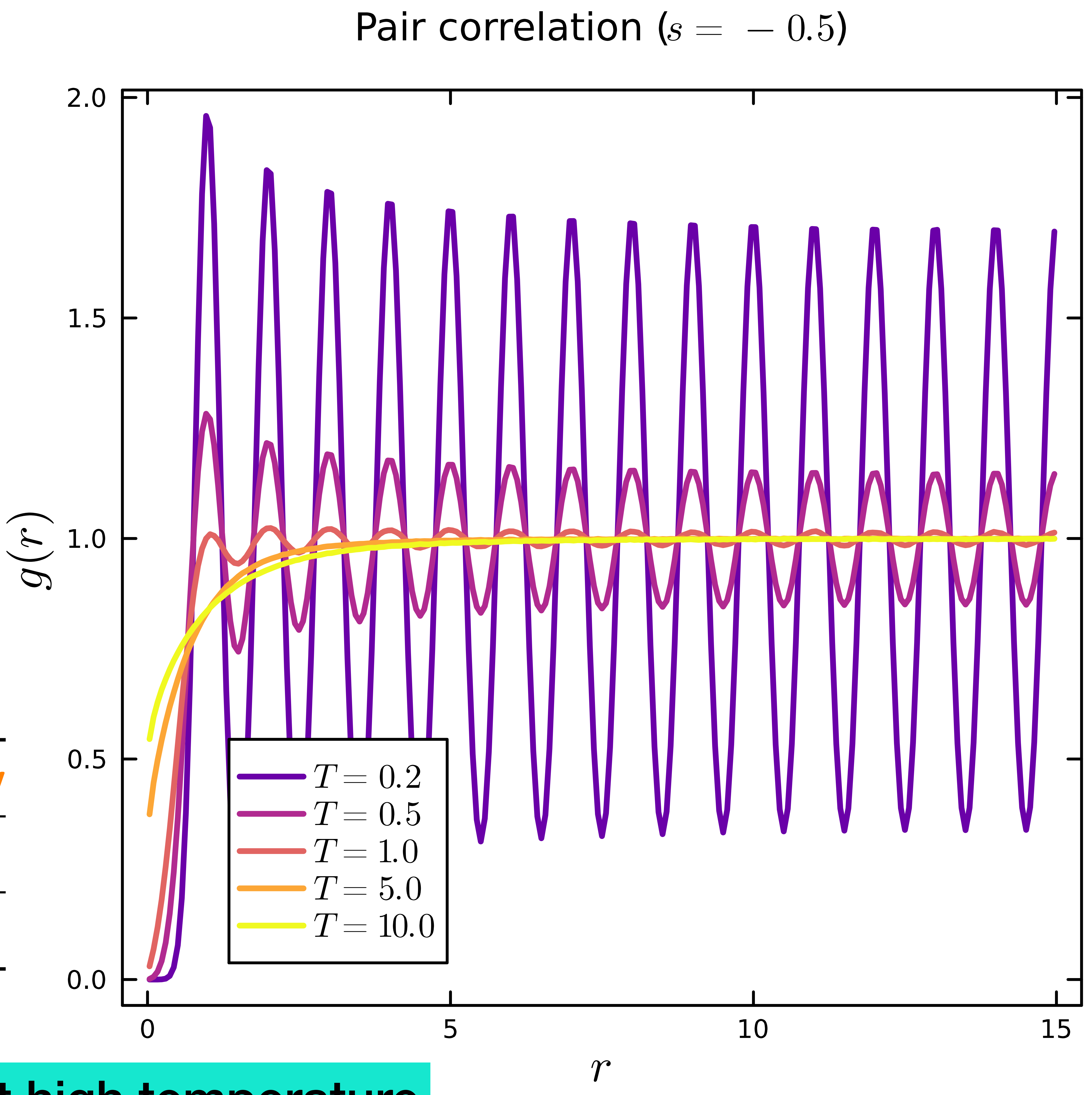
We use periodic boundary conditions $\Omega = \mathbb{Z}/N\mathbb{Z}$

We use **two-point correlation** as detection tools

$\rho^{(2)}(x, y) =$ probability there is a particle at x and another particle at y

By periodicity, we consider $g(r) := \rho^{(2)}(0,r)$

Fluid	$g(r) \to 1$ as $r \to \infty$ **rapidly/monotonically**
Solid	$g(r) \not\to 1$ & is a **periodic** function
Quasisolid (BKT)	$g(r) \to 1$ "very" slowly



Observation — solid at low temperature and fluid at high temperature

Monte Carlo simulations for $-1 < s < 0$ (II)

Another important quantity is the structure factor $S(k) := \widehat{g-1}(k)$

Behavior of S near $k \sim 0$ is linked with behavior of $g(r)$ as $r \to \infty$

Theorem – *For a translation-invariant point process Γ on $\mathbb{R}$, if $S(k) \sim |k|^\eta$ for $\eta > 1$ as $k \sim 0$, then there exists a family of mutually singular (periodic) point processes $\{\Gamma_\theta : \theta \in [0,1]\}$ s.t.*

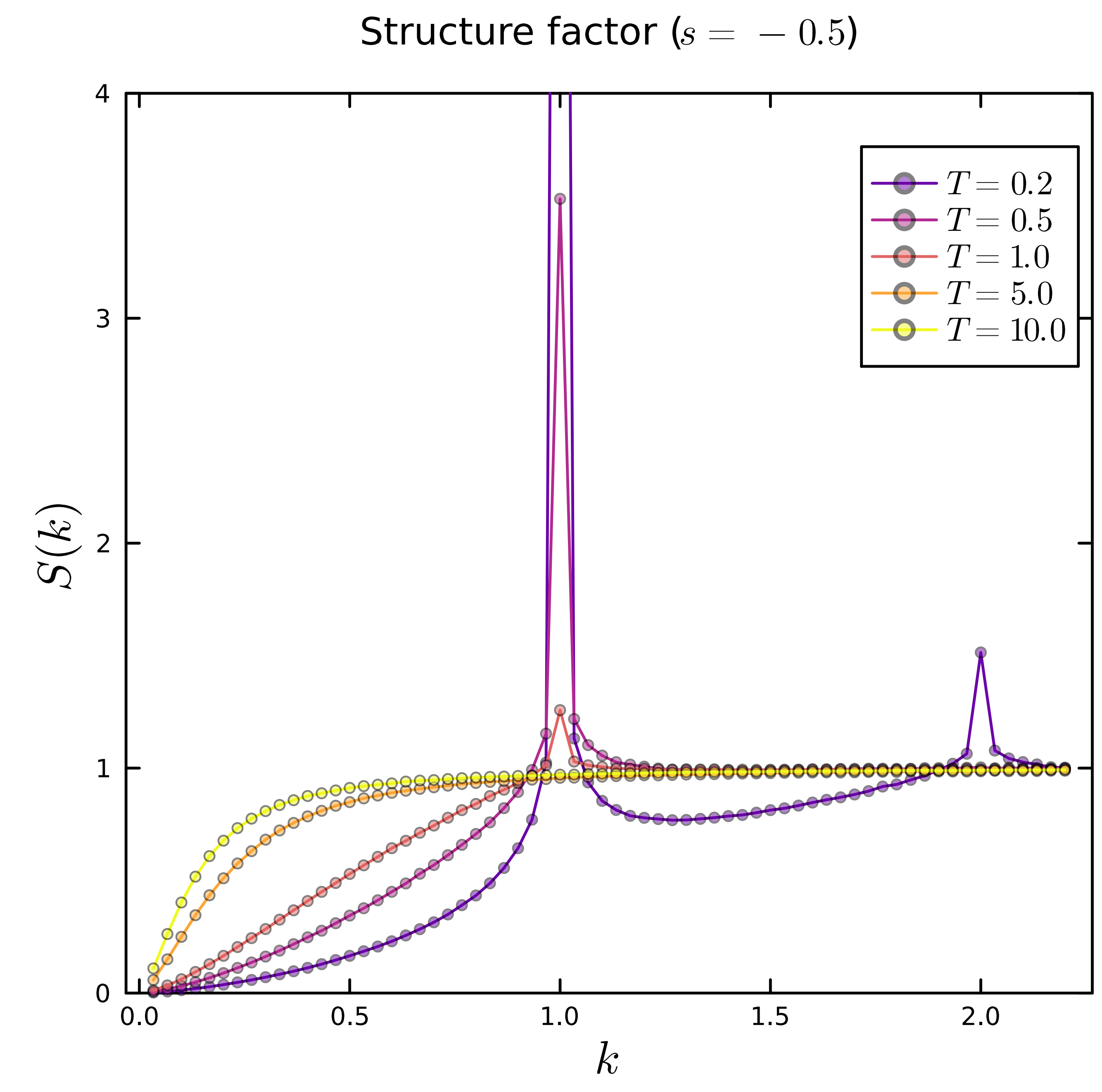
$$\Gamma(\omega) = \int_0^1 \Gamma_\theta(\omega) d\theta$$

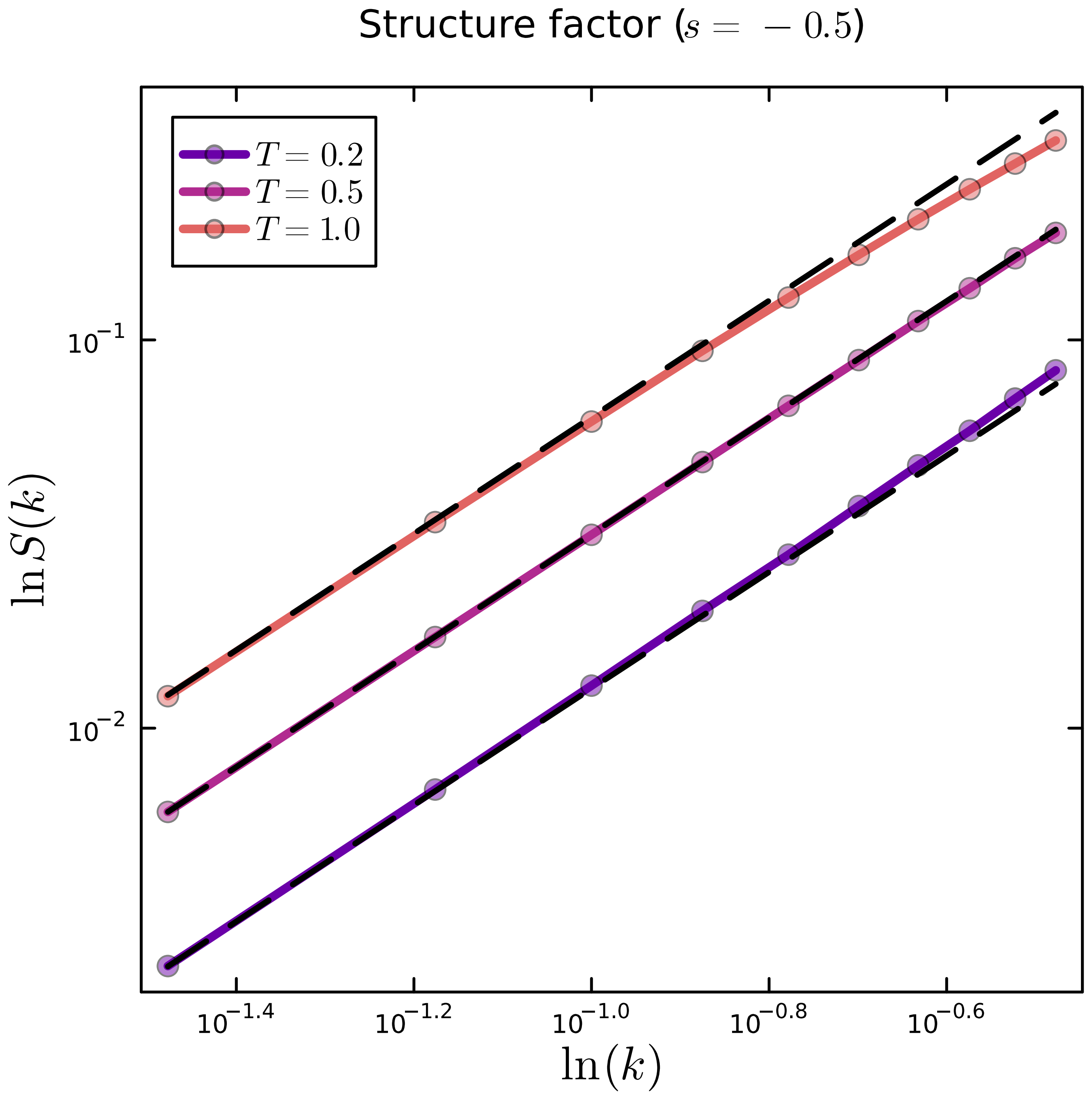
[Aizenman, Goldstein & Lebowitz '01]

In a nutshell : Γ *is obtained by averaging a lattice (i.e. solid)*

It is observed numerically that for small T we have $S(k) \simeq 2T|k|^{1-s}$

Conjecture – *For all $-1 < s < 0$, there exists a threshold T_s such that, for all $T < T_s$ the structure factor $S(k)$ behaves like $S(k) \sim 2T|k|^{1-s}$ when $k \sim 0$. This would imply crystallization by the previous theorem.*





Conjectures and conclusion

From the numerics experiments, we may conjecture :

Conjecture – *For all* $-1 < s < 0$*, there exists a threshold* T_s *such that, for all* $T > T_s$ *the set of all Riesz point process* $\mathscr{R}_{s,T}$ *is reduced to a point which is translation-invariant (i.e. fluid), and such that for* $T < T_s$ *we have the set* $\mathscr{R}_{s,T}$ *corresponds to the uniform probabilities over translations of the periodic lattice* $\rho^{-1}\mathbb{Z}$ *(i.e solid). Moreover,* $s \mapsto T_s$ *is monotonic and* $\lim_{s \to -1} T_s = \infty$ *and* $\lim_{s \to 0} T_s = 0$*.*

In fact, before we can even look for a (partial) proof this conjecture, one needs to define properly the Riesz point processes for $-1 < s < 0$:

Problem – *Can we actually prove that the set* $\mathscr{R}_{s,T}$ *of Riesz point process is non-empty for* $-1 < s < 0$ *in* $d = 1$ *for all* $T \geq 0$ *?*

Non-trivial questions that will hopefully interest an ever-growing community :)

Thank you !

Hyperuniformity

Definition — *A point process Γ is said to be hyperuniform for all bounded $D \subset \mathbb{R}^d$, we have that $Var(n_D) = o(|D|)$ where $n_D(\omega) = \sharp\, \Gamma(\omega) \cap D$.*

The original theorem of [Aizenman, Goldstein & Lebowitz '01] actually reads:

Theorem – *Any translation-invariant point process Γ on $\mathbb{R}$ with bounded fluctuations on number of points, that is $Var(n_D) = O(1)$, then there exists a family of mutually singular…*

Therefore, one way to prove our conjecture is to show that at low enough temperature $T \ll 1$ a Riesz point process for $-1 < s < 0$ has bounded fluctuations of the local number of points.

In fact, if the Riesz point process is indeed a fluid at high enough temperature, we conjecture:

Conjecture – *For all $-1 < s < 0$, there exists a threshold T_s such that for all $T > T_s$ we have $Var(n_D) = O(\ln|D|)$ and for all $T < T_s$ we have $Var(n_D) = O(1)$*

The Berezinsky—Kosterlitz— Thouless transition

It is believed that the 1D log-gas ($s = 0, d = 1$) has no phase transition for $T > 0$

Nevertheless, it exhibits a « phase transition » reminiscent of the BKT transition in $d = 2$

$$g(r) \sim_{r\to\infty} \begin{cases} 1 - \dfrac{T}{\pi^2 r^2} & T > 1/2 \\ 1 + \dfrac{\cos(2\pi r)}{2\pi^2 r^2} - \dfrac{1}{2\pi^2 r^2} & T = 1/2 \\ 1 + c\dfrac{\cos(2\pi r)}{r^{4T}} - \dfrac{1}{2\pi^2 r^2} & T < 1/2 \end{cases}$$

At $T = 1/2$ we transition from an universal decay to one which depends on the temperatures (and oscillating terms appear)

[Forrester '84]

Remark — The expansion entails that

$$S(k) \sim 2T|k| \quad \text{as } k \sim 0$$

$$S(k) \sim |k-1|^{4T-1} \quad \text{as } k \sim 1$$

which can be verified numerically: