PRACTICAL WORK II

You are allowed to write your code in *any* langage that suits you the best, as long as the final code is runnable and debugged. That being said, as previously, a notebook in `Python` is available on my personal webpage at `https://www.ceremade.dauphine.fr/~lelotte/`. The notebook already contains most (if not *all*) of the code needed to answer all the questions of this practical work is a (very) reasonable amount of time — your job is simply to « fill the gaps » in the code. When asked to « comment » or « explain » something, add either a comment (in the code) or a textual cell (in the notebook). Send your work at lelotte@ceremade.dauphine.fr.

I — « Primum prima »

The backbone of SDE's theory is the very existence of the *Wiener process* $(W_t)_{t\geqslant 0}$, named after Norbert Wiener (1894–1964), as well as that of the *Itō integral*, named after Kiyoshi Itō (1915–2008), *i.e.* a rigorous definition of the quantity

$$\int_0^T f(s, X_s)\mathrm{d}W_s. \tag{1}$$

Problem 1. Implement a function `W(T,N)` that (approximatively) simulates the *Wiener process* on $[0, T]$ with step-size $h = T/N$, where $N \in \mathbb{N}$, and plot several paths — *e.g.* with $T = 2$ and $N = 400$. Then, implement a function `I(f, T, N, Ws)` which approximates the Itō integral (1) on $[0, T]$, where `f` is the integrand and `Ws` is a path of the Wiener process.

II — Schemes and orders of convergence

When resorting to a numerical scheme to solve a SDE of the form

$$\mathrm{d}X_t = a(t, X_t)\mathrm{d}t + b(t, X_t)\mathrm{d}W_t, \quad X_0 = x_0, \tag{2}$$

we have seen that, similarly to the deterministic case, one can define the order of strong (resp. weak) convergence $\gamma_s > 0$ (resp. $\gamma_w > 0$) of the considered scheme (see **[Definition 4.19 & 4.20]** of the lecture notes). The **[Theorem 4.22]** states that $\gamma_s = \frac{1}{2}$ and $\gamma_w = 1$ (resp. $\gamma_s = \gamma_w = 1$) for the *Euler-Marumaya scheme* (resp. the *Milstein scheme*).

Problem 2. In this problem, let us consider an important growth model with added noise, namely the *Gompertzian Stochastic Model*,

$$\mathrm{d}X_t = -\beta X_t \ln(X_t)\mathrm{d}t + \gamma X_t \mathrm{d}W_t \quad \text{with } \beta, \gamma \in \mathbb{R}, \tag{3}$$

which corresponds in (2) to the choices $a(t,x) = -\beta x \ln(x)$ and $b(t,x) = \gamma x$. In turns out that the solution $(X_t)_{t\geqslant 0}$ of (3) has an explicit formula, namely **[Bonus — *Prove this formula using the Itō formula by considering* $Y_t = e^{bt}\ln(X_t)$]**.

$$X_t = \exp\left\{ \ln(x_0)e^{-bt} - \frac{c^2}{2b}(1-e^{-bt}) + ce^{-bt}\int_0^t e^{bs}\mathrm{d}W_s \right\}. \tag{4}$$

Implement the *Euler-Maruyama scheme* (resp. the *Milstein scheme*) on $[0,T]$ with $T=1$ and with a uniformly-spaced discretization grid (*i.e.* $h = T/N$ for some $N \in \mathbb{N}$), and confirm (or infirm) numerically their orders of strong (and weak) convergence — comment on the results obtained.

Bonus — Sampling via Unadjusted Langevin Algorithm

Given some probability density $P(\mathbf{x})$ on $\mathbb{R}^d$, an important practical problem is to *sample* from $P(\mathbf{x})$ (*i.e.* draw samples $\mathbf{x}_1, \ldots, \mathbf{x}_N$ distributed according to $P(\mathbf{x})$). For instance, if one wants to approximate the quantity $\mathbb{E}_{X\sim P(\mathbf{x})}[f(X)]$ for some function $f : \mathbb{R}^d \to \mathbb{R}$ through the use of *Monte-Carlo methods*, then one is required to know how to sample from $P(\mathbf{x})$.

Let us suppose that there exists $V : \mathbb{R}^d \to \mathbb{R}$ and $\beta > 0$ such that

$$P(\mathbf{x}) = \left(\int_{\mathbb{R}^d} e^{-\beta V(\mathbf{x})}\mathrm{d}\mathbf{x} \right)^{-1} e^{-\beta V(\mathbf{x})} \tag{5}$$

Probability densities of this form are ubiquitous in applied mathematics — we cannot stress this enough. It turns out that, in this case, $P(\mathbf{x})$ is the unique *invariant measure*[1] to the following SDE, the so-called *(overdamped) Langevin equation*:

$$\mathrm{d}X_t = -\nabla V(X_t)\mathrm{d}t + \sqrt{\frac{2}{\beta}}\mathrm{d}W_t. \tag{6}$$

Moreover (under appropriate hypotheses), the solution to Equation (6) enjoys the following property that, as $t \to \infty$ (and for any initial *datum* x_0), the distribution of X_t « approaches » that of $p(\mathbf{x})$. Therefore, if one can simulate the solution $(X_t)_{t\geqslant 0}$ to Equation (6) for a long enough time, one can effectively sample from $P(\mathbf{x})$.

[1] If X_0 is distributed along $P(\mathbf{x})$, *i.e.* $\mathrm{Law}(X_0) \sim P$, then the solution X_t to Equation (6) verifies $\mathrm{Law}(X_t) \sim P$ for all $t \geqslant 0$.

This leads, for instance, to the following procedure known as the *Unadjusted Langevin Algorithm* (ULA), which reads as follows. Fix a step-size $\tau \ll 1$ and an initial *datum* $x_0 \in \mathbb{R}^d$. For $k = 1, \dots, K$ (where $K \gg 1$) using the *Euler-Marumaya method*, compute the approximation $\widetilde{X}_k$ of $X_{k\tau}$, where X_t is the true solution to Equation (6), that is

$$\widetilde{X}_k = \widetilde{X}_{k-1} - \tau \nabla V(\widetilde{X}_{k-1}) + \sqrt{\frac{2}{\beta}}\xi_k \quad \text{where } \xi_k \sim N(0, \tau I_d). \tag{7}$$

Then, according to what precedes, $\widetilde{X}_K$ is an « approximate » sample of $P(\mathbf{x})$

Problem 3. Given some $v_{\text{ext}} : \mathbb{R} \to \mathbb{R}$, let us consider the probability density $P_{\beta,N}$ on $\mathbb{R}^n$, the so-called (one-dimensional) β-*ensemble* — this terminology pertains to *Random Matrix Theory* — defined as

$$P_{\beta,n}(x_1, \dots, x_n) \propto \prod_{1 \leqslant i < j \leqslant n} |x_i - x_j|^\beta \times \prod_{i=1}^{n} e^{-\beta v_{\text{ext}}(x_i)}. \tag{8}$$

The density $P_{\beta,N}$ relates to *statistical physics*, as it models n electrons roaming on the line $\mathbb{R}$ together with the potential landscape V_{ext}, at temperature $\frac{1}{\beta}$ and interacting with each others through the two-dimensional logarithmic *Coulomb potential*. It turns out — and this is an *extremely* beautiful and *non-trivial* fact — that, when $v_{\text{ext}}(x) = \frac{x^2}{2}$ (*i.e.* v_{ext} is the *harmonic potential*), one can *exactly* sample from (8). In fact, as proved by Dumitriu & Edelman in [1], if H_β is the $n \times n$ random matrix defined as

$$H_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & & \\ & \ddots & \ddots & \ddots & & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta & \\ & & & \chi_\beta & N(0,2) \end{pmatrix} \tag{9}$$

where χ_γ denotes the χ *law* of parameter $\gamma > 0$, then the eigenvalues $\lambda_1, \dots, \lambda_n$ of H_β are distributed along $P_{\beta,N}$, that is

$$(\lambda_1, \dots, \lambda_n) \sim P_{\beta,n}. \tag{10}$$

Question: Implement the ULA algorithm, and confirm its convergence using $P_{\beta,N}$ from which we know how to sample exactly. [2]

[2] If you're done with this question, come to me, I have plenty of other questions regarding this !

References

[1] Dumitriu, I., and Edelman, A. Matrix models for beta ensembles. *Journal of Mathematical Physics 43*, 11 (2002), 5830–5847.